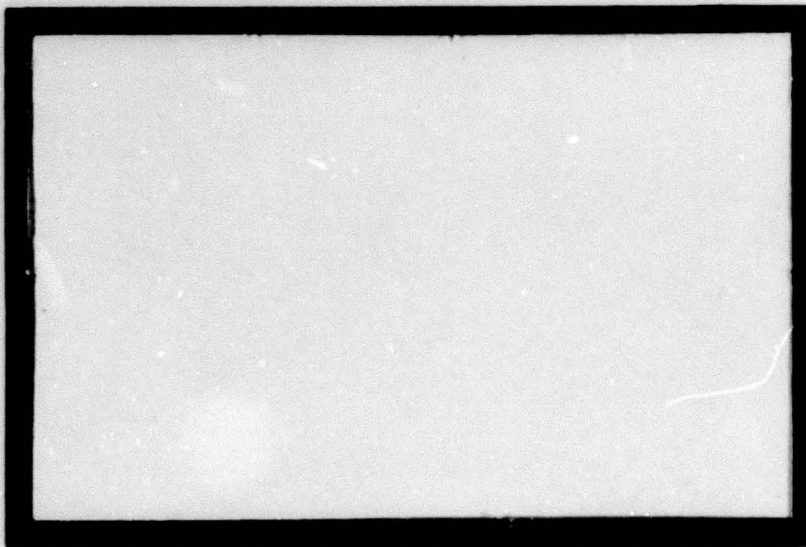
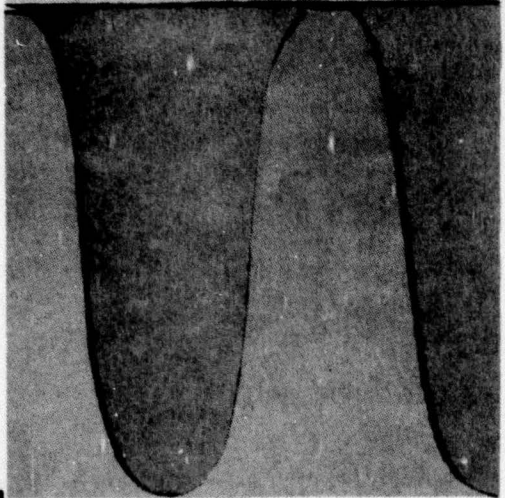


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A TEST OF FIT FOR CONTINUOUS DISTRIBUTIONS  
BASED ON GENERALIZED MINIMUM CHI-SQUARE

John Gurland and Ram C. Dahiya

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# ABSTRACT

A test of fit based on minimum chi-square techniques is developed for continuous distributions. This procedure is investigated in detail for the special case of testing for normality, where the test statistic is based on the first four sample moments. The asymptotic non-null distribution of the general test statistic is obtained, and in particular the power of the test of normality is derived for several alternative families of distributions.

# A TEST OF FIT FOR CONTINUOUS DISTRIBUTIONS BASED ON GENERALIZED MINIMUM CHI-SQUARE

John Gurland and Ram C. Dahiya

## 1. Introduction

In this paper a test of fit for continuous distributions is developed based on generalized minimum chi-square techniques. Although the Pearson chi-square test of fit is widely used especially in the case of discrete distributions, there are difficulties in applying it, particularly in the case of continuous distributions. A discussion of these difficulties is included in the paper by Dahiya and Gurland (1970a). A motivation for obtaining the results in the present paper is to develop a test which is free of complications associated with the Pearson chi-square test. In particular the question of how to form class intervals does not arise in the test of fit presented here. Furthermore the asymptotic distribution is exactly that of a  $\chi^2$ , in contradistinction to the asymptotic distribution of the statistic employed in the Pearson  $\chi^2$  test when the estimators of parameters are obtained from the ungrouped sample (cf. Chernoff and Lehmann (1954)).

The asymptotic non-null distribution of the test statistic proposed here is developed for general alternatives. As a special case the asymptotic power is obtained for testing normality against several specific alternative families of distributions. The power of this test is compared with that of a modified form of the Pearson chi-square test based on random intervals presented by Dahiya and Gurland (1970a, 1970b).

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Although the test of fit presented here is for continuous distributions, the method based on minimum chi-square techniques is quite general and can in fact be adapted to discrete distributions. Hinz and Gurland (1970) have applied such techniques to develop a test of fit for the negative binomial and other contagious distributions.

## 2. Formulation of a test statistic based on sample moments

First we consider the problem in a general context and show how to construct a statistic for testing the fit of a hypothesized distribution based on a set of sample moments. In a subsequent section the result obtained here will be applied to develop a test of normality.

Let  $X_1, X_2, \dots, X_n$  be a random sample from a certain distribution with p.d.f.

$$p_X(x | \theta) \quad (2.1)$$

where  $\theta$  is a parameter vector of  $q$  components, that is

$$\theta' = [\theta_1, \theta_2, \dots, \theta_q] \quad (2.2)$$

Denote the  $j^{\text{th}}$  raw sample moment by

$$m'_j = \frac{1}{n} \sum_{i=1}^n X_i^j \quad (2.3)$$

Let

$$m' = [m'_1, m'_2, \dots, m'_s] \quad (2.4)$$

where  $s$ , ( $s > q$ ), is a fixed number that remains to be specified. (A low value of  $s$  is generally desirable due to the large sampling fluctuations of higher order moments.) Under the assumption that the  $(2s)^{\text{th}}$  order moment of  $X$  exists, we can easily show by making use of the Central Limit Theorem that the asymptotic distribution of  $\sqrt{n}(m-\mu)$  is normal,

$$N(0; G) \quad (2.5)$$

where the vector  $\mu$  is the population counterpart of  $m$ , given by

$$\mu' = [\mu'_1, \mu'_2, \dots, \mu'_s] \quad (2.6)$$

and the  $s \times s$  covariance matrix  $G$  is given by

$$G = (g_{ij}) = (\mu'_{i+j} - \mu'_i \mu'_j) \quad (2.7)$$

If  $h_1, h_2, \dots, h_s$  be  $s$  functions of  $m$ , that is

$$h_i = h_i(m'_1, m'_2, \dots, m'_s) \quad i = 1, 2, \dots, s \quad (2.8)$$

such that their population counterparts

$$\zeta_i = h_i(\mu'_1, \mu'_2, \dots, \mu'_s) \quad i = 1, 2, \dots, s \quad (2.9)$$

are differentiable to the second order with respect to  $\mu'_1, \mu'_2, \dots, \mu'_s$ , then the asymptotic distribution of

$$\sqrt{n}(h-\zeta) \quad (2.10)$$

is given by

$$N(0; \Sigma) \quad (2.11)$$

where

$$\left. \begin{aligned} h' &= [h_1, h_2, \dots, h_s] \\ \zeta' &= [\zeta_1, \zeta_2, \dots, \zeta_s] \\ \Sigma &= JGJ' \end{aligned} \right\} \quad (2.12)$$

and  $J$  is the  $s \times s$  Jacobian matrix  $\left( \frac{\partial \zeta_j}{\partial \mu_i} \right)$ .

From this result it follows that the asymptotic distribution of

$$Q = n(h - \zeta)' \Sigma^{-1}(h - \zeta) \quad (2.13)$$

is that of  $\chi_s^2$ . Furthermore, if  $\hat{\Sigma}$  is a consistent estimator of  $\Sigma$ , which is obtained from  $\Sigma$  on replacing parameters by maximum likelihood or some other consistent estimators, then according to Gurland (1948), Barankin and Gurland (1951), the asymptotic distribution of

$$Q^* = n(h - \zeta)' \hat{\Sigma}^{-1}(h - \zeta) \quad (2.14)$$

is the same as the asymptotic distribution of  $Q$ .

Now suppose we select functions  $h_i$  such that  $\zeta_i$  are linear functions of the parameters  $\theta_1, \theta_2, \dots, \theta_q$ , that is

$$\zeta = W\theta \quad (2.15)$$

where  $W$  is a  $s \times q$  matrix of known constants. In such a case we can find an estimator for  $\theta$  by minimizing the expression for  $Q^*$  in (2.14). This estimator,  $\hat{\theta}$ , say, is given by

$$\hat{\theta} = (W' \hat{\Sigma}^{-1} W)^{-1} W' \hat{\Sigma}^{-1} h. \quad (2.16)$$

Let

$$\hat{\xi} = W \hat{\theta} \quad (2.17)$$

and

$$\hat{Q} = n(h - \hat{\xi})' \hat{\Sigma}^{-1} (h - \hat{\xi}). \quad (2.18)$$

Now let

$$\left. \begin{aligned} \hat{R} &= W(W' \hat{\Sigma}^{-1} W)^{-1} W' \hat{\Sigma}^{-1} \\ \hat{A} &= \hat{\Sigma}^{-1} (I - \hat{R}) \end{aligned} \right\} \quad (2.19)$$

Then

$$\begin{aligned} \hat{Q} &= n(h - \hat{R}h)' \hat{\Sigma}^{-1} (h - \hat{R}h) = nh'(I - \hat{R}) \hat{\Sigma}^{-1} (I - \hat{R})h \\ &= nh' \hat{A} h. \end{aligned} \quad (2.20)$$

From results of Gurland (1948), Barankin and Gurland (1951), the asymptotic distribution of  $nh' \hat{A} h$  is the same as the asymptotic distribution of  $nh' A h$  where  $A$  is obtained from  $\hat{A}$  on replacing  $\hat{\Sigma}$  by  $\Sigma$ . In order to find the distribution of  $nh' A h$  we make use of the following lemma.



Lemma 1.

If  $X$  is distributed as  $N(\mu; \Sigma)$  and  $B$  is a matrix such the  $\Sigma B$  is idempotent then the distribution of  $X'BX$  is non-central chi-square with  $r$  degrees of freedom and noncentrality parameter  $\lambda$ , denoted by  $\chi^2_{r, \lambda}$ , where  $r$  is the rank( $B$ ) and  $\lambda = \mu'B\mu$ .

Proof:

Let  $P$  be a nonsingular matrix such that

$$P\Sigma P' = I, \quad (2.21)$$

an identity matrix. On making use of the transformation

$$Y = PX \quad (2.22)$$

it follows that  $Y$  is distributed as  $N(P\mu; I)$  and

$$X'BX = Y'P'^{-1}BP^{-1}Y. \quad (2.23)$$

Now  $P'^{-1}BP^{-1}$  is an idempotent matrix of rank  $r$  since  $\Sigma B = P^{-1}P'^{-1}B$  is idempotent of rank  $r$ . Hence the distribution of  $X'BX$  is  $\chi^2_{r, \lambda}$  where

$$\lambda = (P\mu)'P'^{-1}BP^{-1}(P\mu) = \mu'B\mu \quad (2.24)$$

This proves Lemma 1.

From the above lemma and also assuming  $W$  of full rank  $q$ , we see that the asymptotic null distribution of  $nh'A_h$  is  $\chi^2_{s-q}$  since  $\Sigma A$  is an idempotent matrix of rank  $s - q$  and  $\xi'A\xi = 0$  which can easily be verified. Thus it follows that the asymptotic distribution of  $\hat{Q}$  is that of  $\chi^2_{s-q}$ .

The statistic  $\hat{Q}$  can be utilized for testing the fit of an assumed distribution. In order to ascertain how well such a test of fit behaves, its power against specific alternatives can be obtained from the non-null distribution given in section 3.

### 3. Asymptotic non-null distribution of $\hat{Q}$

The asymptotic non-null distribution of  $\hat{Q}$  turns out to be that of a weighted sum of independent non-central  $\chi^2$  random variables each with one degree of freedom. A derivation of this result along with the precise weights and non-centralities is given in the following theorem.

#### Theorem 1.

Let the null and alternative hypotheses  $H_0, H_1$  respectively be as follows:

$$H_0: X \text{ has p.d.f. } p_X(x | \theta) \quad (3.1)$$

$$-\infty < x < \infty, \quad \theta' = [\theta_1, \theta_2, \dots, \theta_q]$$

$$H_1: X \text{ has p.d.f. } p_X^{(1)}(x | \gamma) \quad (3.2)$$

$$-\infty < x < \infty, \quad \gamma' = [\gamma_1, \gamma_2, \dots, \gamma_p]$$

where  $p \geq q$ . Here  $\theta$  and  $\gamma$  are parameter vectors.

Then the asymptotic non-null distribution of  $\hat{Q}$ , defined in (2.20), is of the form

$$\sum_{i=1}^{s-q} d_i \chi_{1, a_i}^2. \quad (3.3)$$

The constants  $d_1$  are given by (3.7) and  $a_1$  by (3.8), (3.10).

Proof:

Let us denote the matrix to which  $\hat{\Sigma}$  converges under  $H_1$  in probability by  $\Sigma^*$ , that is

$$\hat{\Sigma} \xrightarrow{P} \Sigma^* \text{ under } H_1. \quad (3.4)$$

Then  $\Sigma^*$  involves the parameter vector  $\gamma$ . Now the asymptotic non-null distribution of  $nh'\hat{A}h$  is the same as that of

$$Q^{(1)} = nh'A^*h \quad (3.5)$$

where  $A^*$  is obtained from  $\hat{A}$  on replacing  $\hat{\Sigma}$  by  $\Sigma^*$ . Let  $\Sigma^{(1)}$  denote the asymptotic covariance matrix of  $\sqrt{n}h$  under  $H_1$  which can be found in the same way as  $\Sigma$  is found under  $H_0$ . Also if  $\zeta^{(1)}$  denotes the population counterpart of  $h$  under  $H_1$ , then the asymptotic non-null distribution of  $\sqrt{n}(h - \zeta^{(1)})$  is that of  $N(0; \Sigma^{(1)})$ . There exists a non-singular matrix  $T$  and an orthogonal matrix  $P$  such that

$$T\Sigma^{(1)}T' = I \quad (3.6)$$

and

$$PT'^{-1}A^*T^{-1}P' = D = \begin{bmatrix} d_1 & & & & & \\ & d_2 & & & & \\ & & d_3 & & & \\ & & & \ddots & & \\ & & & & d_{s-q} & \\ & & & & & 0 \\ & & & & & & 0 \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{bmatrix} \quad (3.7)$$

where  $I$  is the identity matrix and  $D$  is a diagonal such that the last  $q$  diagonal elements of  $D$  are zero. This is possible since  $\text{rank}(A^*) = s - q$ .

Let

$$\left. \begin{aligned} u &= PTh \\ \psi &= PT\zeta^{(1)} \end{aligned} \right\} \quad \text{and} \quad (3.8)$$

Then we have

$$\begin{aligned} nh'A^*h &= n(T^{-1}P'u)'A^*(T^{-1}P'u) = nu'Du \\ &= \sum_1^{s-q} d_i (\sqrt{n} u_i)^2. \end{aligned} \quad (3.9)$$

Now since the asymptotic distribution of  $\sqrt{n}(h - \zeta^{(1)})$  is  $N(0; \Sigma^{(1)})$ , it follows that the asymptotic distribution of  $\sqrt{n}(u - \psi)$  is  $N(0; I)$ .

Hence the asymptotic distribution of  $nh'A^*h$  is that of

$$\sum_{i=1}^{s-q} d_i x_{1,a_i}^2$$

where

$$a_i = \sqrt{n} \psi_i \quad i = 1, \dots, s-q \quad (3.10)$$

and  $\psi_i$  is the  $i^{\text{th}}$  element of  $\psi$ . This proves the theorem.

#### 4. Test of fit for normal distribution based on $\hat{Q}$

We shall now consider a test of fit based on  $\hat{Q}$  when the null distribution is normal, that is,  $X$  has p.d.f.

$$p_X(x | \theta) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x-\theta_1)^2}{2\theta_2}} \quad (4.1)$$

$$-\infty < x < \infty, \quad -\infty < \theta_1 < \infty, \quad \theta_2 > 0.$$

Let  $m_2$ ,  $m_3$  and  $m_4$  be second, third and fourth central sample moments respectively. The statistics  $b_1$ ,  $b_2$  given by

$$b_1 = m_3/m_2^{3/2}, \quad b_2 = m_4/m_2^2 \quad (4.2)$$

are sometimes employed for testing normality by means of skewness and kurtosis. Instead of considering these two statistics separately it appears more rational to formulate a single statistic involving the first four moments. This motivates our selection of functions  $h_i$  based on the first four sample moments. The mean, variance, third and fourth central moments of  $X$  are respectively given by:

$$\mu_1' = \theta_1, \mu_2 = \theta_2, \mu_3 = 0, \mu_4 = 3\theta_2^2 \quad (4.3)$$

If we define

$$\left. \begin{aligned} \theta_2^* &= \log \theta_2 \\ \zeta' &= [\mu_1', \log \mu_2, \mu_3, \log(\frac{\mu_4}{3})] \end{aligned} \right\} \quad (4.4)$$

then the elements of  $\zeta$  are linear functions of the parameters  $\theta_1$  and  $\theta_2^*$ .

We can now write

$$\zeta = W\theta^* \quad (4.5)$$

with

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad \theta^* = \begin{bmatrix} \theta_1 \\ \theta_2^* \end{bmatrix}$$

The corresponding  $h_i$  functions are given by

$$h_1 = m_1'; \quad h_2 = \log m_2; \quad h_3 = m_3; \quad h_4 = \log(\frac{m_4}{3}) \quad (4.6)$$

where  $m_1'$  is the sample mean and  $m_2, m_3, m_4$  denote second, third and fourth central sample moments, respectively, as previously indicated.

The transformation from sample raw moments to functions  $h_i$  is achieved in two stages, that is, from  $[m_1', m_2', m_3', m_4']$  to  $[m_1', m_2, m_3, m_4]$  and then finally to  $[h_1, h_2, h_3, h_4]$ . In the notations of section 3,  $\sqrt{n}(h - \zeta)$  is asymptotic  $N(0; \Sigma)$ , where

$$\Sigma = J_2 J_1 G J_1' J_2' \quad (4.7)$$

$$J_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3\theta_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad J_2' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/(3\theta_2^2) \end{bmatrix} \quad (4.8)$$

$$G = \begin{bmatrix} \theta_2 & 0 & 3\theta_2^2 & 0 \\ 0 & 2\theta_2^2 & 0 & 12\theta_2^3 \\ 3\theta_2^2 & 0 & 15\theta_2^3 & 0 \\ 0 & 12\theta_2^3 & 0 & 96\theta_2^4 \end{bmatrix} \quad (4.9)$$

After simplification we obtain:

$$\Sigma = \begin{bmatrix} \theta_2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 6\theta_2^3 & 0 \\ 0 & 4 & 0 & 32/3 \end{bmatrix} \quad (4.10)$$

Now let

$$\hat{\Sigma} = [\Sigma]_{\theta_2 = m_2} \quad (4.11)$$

where  $m_2$  is the maximum likelihood estimator of  $\theta_2$ . Then a statistic  $\hat{Q}$  for testing normality is given by

$$\hat{Q} = nh' \hat{A} h \quad (4.12)$$

where

$$\left. \begin{aligned} \hat{A} &= \hat{\Sigma}^{-1}(I - \hat{R}) \\ \hat{R} &= W(W'\hat{\Sigma}^{-1}W)^{-1}W'\hat{\Sigma}^{-1} \end{aligned} \right\} \quad (4.13)$$

After simplification we can show that

$$\hat{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & -.75 \\ 0 & 0 & 1/(6m_2^3) & 0 \\ 0 & -.75 & 0 & .375 \end{bmatrix} \quad (4.14)$$

Hence a simplified form of  $\hat{Q}$  is given by

$$\hat{Q} = nu'\hat{B}u \quad (4.15)$$

where

$$u' = [h_2, h_3, h_4] = [\log(m_2), m_3, \log(\frac{m_4}{3})] \quad (4.16)$$

and

$$\hat{B} = \begin{bmatrix} 1.5 & 0 & -.75 \\ 0 & 1/(6m_2^3) & 0 \\ -.75 & 0 & .375 \end{bmatrix} \quad (4.17)$$

The statistic  $\hat{Q}$  in (4.15) can easily be computed on a desk calculator.



The asymptotic distribution of  $\hat{Q}$  is  $\chi_2^2$  since here  $s = 4$  and  $q = 2$  in the notations of section 2. Thus to carry out a test of fit for normality at a particular level of significance, one merely requires the corresponding critical point of the  $\chi_2^2$  distribution.

##### 5. Power of the test of normality

Let  $p_X^{(1)}(x|\gamma)$ , where  $\gamma' = [\gamma_1, \gamma_2, \dots, \gamma_p]$  is a parameter vector, denote a general alternative to the null hypothesis of normality. If we denote its  $i^{\text{th}}$  raw moment by  $\mu_i^{(1)}$  and the corresponding central moment by  $\mu_i^{(1)}$  then the asymptotic non-null distribution of  $\sqrt{n}(h - \zeta^{(1)})$  is  $N(0, \Sigma^{(1)})$ , where

$$\left. \begin{aligned} \zeta^{(1)'} &= [\mu_1^{(1)'}, \log \mu_2^{(1)}, \mu_3^{(1)}, \log (\frac{\mu_4^{(1)}}{3})] \\ \Sigma^{(1)} &= (J_2^{(1)} J_1^{(1)}) G^{(1)} (J_2^{(1)} J_1^{(1)})' \\ G^{(1)} &= (\mu_{i+j}^{(1)'} - \mu_i^{(1)'} \mu_j^{(1)'}) \quad i, j = 1, 2, 3, 4 \end{aligned} \right\} \quad (5.1)$$

$$J^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_{2,1} & 1 & 0 & 0 \\ a_{3,1} & a_{3,2} & 1 & 0 \\ a_{4,1} & a_{4,2} & a_{4,3} & 1 \end{bmatrix} \quad (5.2)$$

with

$$a_{2,1} = -2\mu_1^{(1)'}$$

$$a_{3,1} = 3[2\mu_1^{(1)'}{}^2 - \mu_2^{(1)'}]$$

$$a_{3,2} = -3\mu_1^{(1)'}$$

$$a_{4,1} = 4(-3\mu_1^{(1)'}{}^3 + 3\mu_1^{(1)'}\mu_2^{(1)'} - \mu_3^{(1)'})$$

$$a_{4,2} = 6\mu_1^{(1)'}{}^2; \quad a_{4,3} = -4\mu_1^{(1)'}$$

$$J_2^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\mu_2^{(1)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\mu_4^{(1)} \end{bmatrix} \quad (5.3)$$

$\Sigma$  and  $\Sigma^{(1)}$  are the asymptotic covariance matrices of  $\sqrt{n}h$  under  $H_0$  and  $H_1$  respectively and it gives an insight about the test on examining how the structure of these two matrices differs depending on the distribution assumed. In fact the sensitivity of the test based on  $\hat{Q}$  will depend on the difference in the structure of these two matrices.

Now we shall prove an attractive invariant property of the test based on  $\hat{Q}$  defined by (4.15).

#### Theorem 2.

The power of the test based on  $\hat{Q}$ , defined by (4.15) is invariant with respect to the location and scale parameters of the alternative distribution.

Proof:

Since  $m_2 \xrightarrow{P} \mu_2^{(1)}$  under  $H_1$ , the asymptotic non-null distribution of  $\hat{Q} = nu'\hat{B}u$  is the same as the asymptotic non-null distribution of

$$Q^{(1)} = nu'B^{(1)}u \quad (5.4)$$

where

$$B^{(1)} = \begin{bmatrix} 1.50 & 0 & -.75 \\ 0 & 1/(\mu_2^{(1)})^3 & 0 \\ -.75 & 0 & .375 \end{bmatrix} \quad (5.5)$$

The distribution of  $u$  is invariant with respect to the location parameter and  $B^{(1)}$  does not involve this parameter, hence it follows that the asymptotic non-null distribution of  $Q^{(1)}$  does not involve the location parameter.

Now let  $\beta$  be the scale parameter in the alternative distribution of  $X$ . If we take

$$Y = \frac{X}{\beta} \quad (5.6)$$

then the distribution of  $Y$  does not involve  $\beta$ .

Let  $V' = [V_1, V_2, V_3]$  be such that

$$\left. \begin{aligned} V_1 &= u_1 - 2 \log \beta = \log m_2(y) \\ V_2 &= u_2/\beta^3 = m_3(y) \\ V_3 &= u_3 - 4 \log \beta = \log m_4(y) - \log 3 \end{aligned} \right\} \quad (5.7)$$

where

$$m_i(y) = m_i/\beta^i, \quad i = 2, 3, 4. \quad (5.8)$$

Then the distribution of  $V$  does not involve the parameter  $\beta$  since the distributions of  $m_2(y)$ ,  $m_3(y)$  and  $m_4(y)$  do not involve this parameter.

If  $\mu_2(Y)$  be the variance of  $Y$  then we have

$$\mu_2^{(1)} = \mu_2(Y)\beta^2 \quad (5.9)$$

and

$$\begin{aligned} Q^{(1)} &= u'B^{(1)}u = [V_1 + 2 \log \beta, V_2\beta^3, V_3 + 4 \log \beta]B^{(1)} \begin{bmatrix} V_1 + 2 \log \beta \\ V_2\beta^3 \\ V_3 + 4 \log \beta \end{bmatrix} \quad (5.10) \\ &= V'B^*V \end{aligned}$$

where

$$B^* = \begin{bmatrix} 1.50 & 0 & -.75 \\ 0 & 1/(6\mu_2^3(Y)) & 0 \\ -.75 & 0 & .375 \end{bmatrix}$$

It is surprising that although the scale parameter  $\beta$  is involved in  $u$  and  $B^{(1)}$ , it cancels out in  $u'B^{(1)}u$  as is evident in (5.10).

Since the distribution of  $V$  does not involve  $\beta$  and since  $B^*$  is also free of this parameter, the asymptotic distribution of  $Q^{(1)}$  and hence that of  $\hat{Q}$  does not involve the scale parameter  $\beta$ . This completes the proof of Theorem 2.

#### 6. Calculation of power for the test of normality based on $\hat{Q}$

For studying the behavior of the test of fit for normality based on  $\hat{Q}$  we have carried out power computations for several alternative families of distributions. The null hypothesis has been stated in (4.1) and the test statistic  $\hat{Q}$  formulated in (4.15).

The following alternative distributions  $A_1, A_2, A_3, A_4, A_5$  are considered.

$A_1$ : Exponential

$$p_X^{(1)}(x | \gamma) = \frac{1}{\gamma_2} e^{-\frac{x-\gamma_1}{\gamma_2}} \quad x \geq \gamma_1$$

$$-\infty < \gamma_1 < \infty, \quad \gamma_2 > 0$$

$A_2$ : Double Exponential

$$p_X^{(1)}(x | \gamma) = \frac{1}{2\gamma_2} e^{-\frac{|x-\gamma_1|}{\gamma_2}} \quad -\infty < x < \infty$$

$$-\infty < \gamma_1 < \infty, \quad \gamma_2 > 0$$

A<sub>3</sub>: Logistic

$$p_X^{(1)}(x | \gamma) = \frac{e^{-\frac{x-\gamma_1}{\gamma_2}}}{\gamma_2 (1 + e^{-\frac{x-\gamma_1}{\gamma_2}})^2} \quad -\infty < x < \infty$$

$$-\infty < \gamma_1 < \infty, \quad \gamma_2 > 0$$

A<sub>4</sub>: Pearson Type III

$$p_X^{(1)}(x | \gamma) = \frac{e^{-\frac{x-\gamma_1}{\gamma_2}}}{\gamma_2 \Gamma(\beta)} \left(\frac{x-\gamma_1}{\gamma_2}\right)^{\beta-1} \quad x > \gamma_1$$

$$-\infty < \gamma_1 < \infty, \quad \gamma_2 > 0, \quad \beta > 0.$$

A<sub>5</sub>: "Power Distribution"

$$p_X^{(1)}(x | \gamma) = \frac{e^{-\frac{1}{2} \left| \frac{x-\gamma_1}{\gamma_2} \right|^{\frac{2}{1+\beta}}}}{\gamma_2 \Gamma(1 + \frac{1+\beta}{2}) 2^{\frac{1+\beta}{2}}} \quad -\infty < x < \infty$$

$$-\infty < \gamma_1 < \infty, \quad \gamma_2 > 0, \quad \beta > -1.$$

All the alternatives A<sub>1</sub> - A<sub>5</sub>, inclusive, involve unknown parameters  $\gamma_1$  and  $\gamma_2$  which are location and scale respectively. Thus the power will be the same for all possible values of  $\gamma_1$  and  $\gamma_2$  according to the result proved in Theorem 2.

The asymptotic power is given by

$$P\left\{\sum_{i=1}^2 d_i x_{1,a_i}^2 \geq \chi_2^2(\alpha)\right\} \quad (6.1)$$

where the asymptotic non-null distribution of  $\hat{Q}$  is that of  $\sum_{i=1}^2 d_i x_{1,a_i}^2$  as proved in Theorem 1, and  $\chi_2^2(\alpha)$  is the 100(1- $\alpha$ ) percent point of the  $\chi_2^2$  distribution.

A generalization of Gurland's (1955, 1956) Laguerre series expansion has been given by Kotz et al (1967) for the distribution of quadratic forms in non-central normal variates. We make use of this expansion in order to compute the power given by (6.1). These calculations have been carried out for sample sizes  $n = 50, 75, 100$ , and the two levels of significance  $\alpha = .05, .01$ . The results appear in Table 1 for all the alternatives  $A_1$  through  $A_5$ , with several different specified values of the parameter  $\beta$  in the case of  $A_4$  and  $A_5$  as indicated in the table.

A modified form of the Pearson chi-square test has been considered by Dahiya and Gurland (1970b) where the test statistic is denoted by  $X_R^2$ . According to this modification, the estimators obtained from the ungrouped sample are utilized in determining the class interval end points as well as in the test statistic  $X_R^2$ . For convenience in making some comparisons with the  $\hat{Q}$  test the values of power of the  $X_R^2$  test against the alternatives listed in Table 1 are included for those cases corresponding to sample sizes  $n = 50, 100$  which are available from Dahiya and Gurland (1970b). These values are enclosed in parentheses and are based in each case on the number of class intervals giving the maximum power in Tables 1, 2 of Dahiya and Gurland (1970b). For example, in the case of alternative  $A_1$  the power of the  $X_R^2$  test attains a maximum value of 1.000 for sample sizes

$n = 50, 100$  when the number of class intervals is 7, and in the case of alternative  $A_2$  its power attains maximum values .547, .800 corresponding to sample sizes 50, 100 respectively based on 3 class intervals.

It is evident on examining the values of power for the  $\hat{Q}$  test in Table 1 that for most of the cases considered there its value is higher, and sometimes very much higher, than the value for the  $X_R^2$  test.

As we examine Table 1 in detail, we note that for alternatives  $A_1$  and  $A_2$ , namely, the exponential and double exponential, the power is rather high. For the exponential, the power is slightly lower than for that of the  $X_R^2$  test whereas for the double exponential, the reverse is true. For a logistic alternative the difference in the power of the two tests is dramatic. For example, when  $n = 100$  the power of the  $X_R^2$  test with optimal number of classes  $k = 3$  is .180 for  $\alpha = .05$  whereas for the  $\hat{Q}$  test the corresponding power is .654.

As regards  $A_4$ , namely the Pearson Type III, it is evident from the table that the power is higher for low values of the parameter  $\beta$  and decreases slowly as  $\beta$  increases. The decrease in the power is explained by the fact that the alternative  $A_4$  tends to normal as  $\beta$  becomes increasingly large. As evident from the few values of power of the  $X_R^2$  test appearing for alternative  $A_4$  it behaves similarly to the  $\hat{Q}$  test although its power is substantially less.

Alternative  $A_5$  is considered in the table with values of  $\beta$  decreasing from 3.0 to  $-.95$ . Similar to the behavior of the  $X_R^2$  test, the power increases as  $\beta$  increases for  $\beta > 0$ , and it also increases as  $\beta$  decreases for  $\beta < 0$ , which behavior is explained by the fact that the normal distribution is a special case of the family  $A_5$  with  $\beta = 0$ . For all the values of  $\beta$  considered here except  $\beta = -.50$ , the power of  $\hat{Q}$  test is obviously higher than that of the  $X_R^2$  test.



TABLE I  
Power of  $\hat{Q}$  test for normality

Alternative	$\alpha = .05$			$\alpha = .01$		
	n = 50	75	100	50	75	100
A <sub>1</sub> : Exponential	.927 (1.000)	.941	.953 (1.000)	.892	.913	.930
A <sub>2</sub> : Double Exponential	.833 (.547)	.858	.879 (.800)	.754	.789	.818
A <sub>3</sub> : Logistic	.606 (.128)	.631	.654 (.180)	.465	.495	.523
A <sub>4</sub> : with $\beta = .5$	.966	.972	.976	.949	.957	.964
A <sub>4</sub> : with $\beta = 2.0$	.865	.898	.923	.804	.849	.883
A <sub>4</sub> : with $\beta = 2.5$	.839 (.502)	.879	.909 (.864)	.765	.820	.861
A <sub>4</sub> : with $\beta = 3.0$	.814 (.391)	.860	.895 (.716)	.732	.791	.837
A <sub>4</sub> : with $\beta = 3.5$	.790 (.318)	.841	.881 (.597)	.698	.762	.814
A <sub>4</sub> : with $\beta = 4.0$	.767 (.268)	.822	.865 (.506)	.667	.734	.789
A <sub>4</sub> : with $\beta = 5.0$	.722 (.205)	.784	.834 (.381)	.608	.680	.741
A <sub>4</sub> : with $\beta = 10.0$	.548	.617	.678	.407	.473	.534
A <sub>5</sub> : with $\beta = 3.0$	.996	.996	.997	.994	.995	.995
A <sub>5</sub> : with $\beta = 2.0$	.974	.976	.978	.960	.963	.966
A <sub>5</sub> : with $\beta = .95$	.815	.842	.866	.730	.767	.799
A <sub>5</sub> : with $\beta = .75$	.721 (.376)	.759	.792 (.603)	.604	.652	.695
A <sub>5</sub> : with $\beta = .50$	.527 (.211)	.571	.611 (.343)	.372	.418	.462
A <sub>5</sub> : with $\beta = .25$	.249	.271	.292	.117	.133	.149
A <sub>5</sub> : with $\beta = -.50$	.036 (.144)	.105	.216 (.262)	.002	.009	.029
A <sub>5</sub> : with $\beta = -.75$	.154	.483	.785	.008	.078	.280
A <sub>5</sub> : with $\beta = -.95$	.328 (.331)	.779	.969 (.583)	.027	.237	.621

n = sample size

$\alpha$  = level of significance

A<sub>4</sub> corresponds to the Pearson Type III distribution

A<sub>5</sub> corresponds to the "power distribution"

## 7. Conclusion

The use of the statistic  $\hat{Q}$  in testing for normality results in high values of power for many of the alternatives considered in Table 1. The form of  $\hat{Q}$  for this test turns out to be relatively simple and could, in fact, be computed on a desk calculator if need be. A modified form of the Pearson chi-square statistic, designated as  $X_R^2$ , which could also be used to test for normality as shown by Dahiya and Gurland (1970a, 1970b) has been compared with the  $\hat{Q}$  test for several cases of the alternatives considered in Table 1 and found to have lower power for the most part.

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